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A Name-free Account of Action Calculi

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Abstract

Action calculi provide a unifying framework for representing a variety of models of communication, such as CCS, Petri nets and the π-calculus, within a unified setting. A central idea is to model the interaction between actions using names. We introduce a name-free account of action calculi, called the closed action calculi, and show that there is a strong correspondence between the original presentation and the name-free presentation. These results show that, although names play an important presentational role, they are in some sense inessential.

1 Introduction

Action calculi, introduced by Milner [10], provide a framework for representing a variety of models of communication, such as CCS [7], Petri nets [14] and the π-calculus [8], within a unified setting. A central idea is to model the interaction of actions using names. We introduce a name-free account of action calculi, called the closed action calculi, and show that there is a strong correspondence between Milner's presentation and the name-free presentation. These results show that, although names play an important presentational role, they are in some sense inessential.

Action calculi provide a uniform account of names, with different sets of constants, called *controls*, specifying different models of computation. Each action calculus $\mathrm{AC}(\mathcal{K})$ can be presented as a quotient of a set of terms, specified by the set of controls $\mathcal{K}$, over an equational theory AC. The equational theory captures the underlying behaviour of the names. Accompanying each action calculus is a reaction relation on the equivalence classes, or *actions*, which accounts for the behaviour of the controls. In this paper, we show that every action calculus $\mathrm{AC}(\mathcal{K})$ has a corresponding closed action calculus $\mathrm{CAC}(\mathcal{K}')$, where $\mathcal{K}'$ is uniquely determined by $\mathcal{K}$. Given a term in $\mathrm{AC}(\mathcal{K})$

[1] The full paper, *Action Calculi VII: Closed Action Calculi*, is in Milner's ftp directory. The author acknowledges support from a BP Royal Society of Edinburgh Research Fellowship.

with free names in $\{\vec{x}\}$, we define the term $[\![t]\!]_{\vec{x}}$ in $\mathrm{CAC}(\mathcal{K}')$, called its *closure*. Given $s = t$ in $\mathrm{AC}(\mathcal{K})$, we have $[\![s]\!]_{\vec{x}} = [\![t]\!]_{\vec{x}}$ in $\mathrm{CAC}(\mathcal{K}')$, whenever the free names in s and t are contained in $\{\vec{x}\}$. We also define an equality-preserving function $\langle\!\langle \_ \rangle\!\rangle$ the other way, and show that

$$\langle\!\langle [\![t]\!]_{\vec{x}} \rangle\!\rangle = (\vec{x})t \in \mathrm{AC}, \quad \text{and}$$

$$[\![\langle\!\langle t \rangle\!\rangle]\!]_{[\,]} = t \in \mathrm{CAC},$$

where $(\vec{x})t$ denotes the abstraction of term t with respect to the list of names $\vec{x}$. These results justify our intuition that $[\![\_]\!]_{\vec{x}}$ can be viewed as a closure function.

A general account connecting the reaction relations of $\mathrm{AC}(\mathcal{K})$ and $\mathrm{CAC}(\mathcal{K}')$ is beyond the scope of this paper. We look at the action calculi representing the π-calculus and the λ-calculus, to illustrate that such a correspondence is possible in these cases; such a correspondence is possible for all the action calculi studied by Milner in [10] and [11]. Finally, we also show that our results easily extend to Milner's reflexive action calculi.

Summary We give an overview of action calculi in section 2 to make the paper self-contained. In section 3, we introduce the corresponding closed action calculi. Sections 4 contains the definitions of $[\![\_]\!]_{\vec{x}}$ and $\langle\!\langle \_ \rangle\!\rangle$ respectively, and the associated results outlined above. Section 5 explores the correspondence between the reaction relations of two action calculi and their corresponding closed action calculi. In section 6, we show how our ideas extend to reflexive action calculi. We conclude with some remarks regarding future research.

2 Action Calculi

We give a brief account of action calculi presented as the quotient of a term algebra [10]. We also give a type-theoretic presentation of action calculi, which gives a local account of names using contexts. This type-theoretic presentation is used to link action calculi with their corresponding closed action calculi, defined in section 3.

An action calculus is defined by a set of terms, an equational theory on terms and a preorder on the equivalence classes, called a *reaction relation*. Each action calculus presupposes a freely generated monoid $(M, \otimes, \varepsilon)$, whose elements are called *arities*, and a denumerable set X of names, to each of which is assigned a prime arity [2]. Unless otherwise stated, the set X of names will remain fixed, as will the monoid $(M, \otimes, \varepsilon)$ and the assignment of arities to names. An action calculus $\mathrm{AC}(\mathcal{K})$ is specified by a set $\mathcal{K}$ of controls, each equipped with an arity rule, together with a set of reaction rules which define its reaction relation. Terms have the form $t : m \to n$, for $m, n \in (M, \otimes, \varepsilon)$, where t is constructed from the basic operators $\mathbf{id}_m, \cdot, \otimes, \mathbf{ab}_x, \boldsymbol{\omega}_p$ and $\langle x \rangle$, and the controls $K \in \mathcal{K}$. We let k, l, m, n range over arities, let p, q denote prime arities, let x, y, z, w range over names, and write $x : k$ to mean that x has arity k.

[2] Since the monoid is freely generated, there exists a set of prime arities which generate the monoid.

Definition 2.1 **[Terms]** The *set of terms* over $\mathcal{K}$, denoted by $\mathrm{T}(\mathcal{K})$, is generated by the following rules:

$$\mathbf{id}_m : m \to m$$

$$\frac{s : k \to l \qquad t : l \to m}{s \cdot t : k \to m}$$

$$\frac{s : k \to m \qquad t : l \to n}{s \otimes t : k \otimes l \to m \otimes n}$$

$$\frac{t : m \to n}{\mathbf{ab}_x(t) : p \otimes m \to p \otimes n}\; x : p$$

$$\omega_p : p \to \epsilon$$

$$\langle x \rangle : \varepsilon \to p, \quad x : p$$

$$* \; \frac{t_1 : m_1 \to n_1 \quad \ldots \quad t_r : m_r \to n_r}{K(t_1, \ldots, t_r) : m \to n}\; \chi$$

where each control operator $K \in \mathcal{K}$ is accompanied by an arity rule $*$, such that side-condition χ may constrain the value of the integer r and the arities m_i, n_i, m and n. Terms of the form $K(t_1, \ldots, t_r)$ are called *control terms.* We shall omit the arity subscripts on the basic operators when they are apparent. The notions of *free name* and *bound name* are standard; $\mathbf{ab}_x$ binds x and $\langle x \rangle$ represents a free occurrence of x. The set of names free in t is denoted by $fn(t)$. We let $fn(s,t)$ denote the set $fn(s) \cup fn(t)$. We let $\mathrm{T}_{\{\vec{x}\}}(\mathcal{K})$ denote the set of terms whose free names are contained in $\{\vec{x}\}$. Given a possibly empty sequence of names $[x_1 : p_1, \ldots, x_r : p_r]$ denoted by $\vec{x}$, we use the notation $|\vec{x}|$ to denote $p_1 \otimes \ldots \otimes p_r$.

Definition 2.2 **[Derived operations]** We define an alternative form of *abstraction* $(x)t$, the *permutations* $\mathbf{p}_{m,n}$, the *copying* operator $\mathbf{copy}_m$ and some other standard abbreviations as follows:

$$\begin{array}{lll}
(x)t & \stackrel{\text{def}}{=} \mathbf{ab}_x(t) \cdot (\omega \otimes \mathbf{id}) & \\
(\vec{x})t & \stackrel{\text{def}}{=} (x_1) \cdots (x_r)t, & (\vec{x} = [x_1, \cdots, x_r], \text{ all distinct, } r \geq 1) \\
\langle \vec{x} \rangle & \stackrel{\text{def}}{=} \langle x_1 \rangle \otimes \cdots \otimes \langle x_r \rangle, & (\vec{x} = [x_1, \cdots, x_r], \; r \geq 1) \\
\mathbf{p}_{m,n} & \stackrel{\text{def}}{=} (\vec{x}, \vec{y})\langle \vec{y}, \vec{x} \rangle, & (|\vec{x}| = m, |\vec{y}| = n) \\
\mathbf{copy}_m & \stackrel{\text{def}}{=} (\vec{x})\langle \vec{x}, \vec{x} \rangle & \\
\omega_m & \stackrel{\text{def}}{=} \begin{cases} \omega_{p_1} \otimes \ldots \otimes \omega_{p_r}, & (m = p_1 \otimes \ldots \otimes p_r) \\ \mathbf{id}_\varepsilon, & (m = \varepsilon) \end{cases} &
\end{array}$$

We assume that $(\,)t$ denotes the term t and $\langle\,\rangle$ denotes the term $\mathbf{id}_\varepsilon$. Notice that $\mathbf{p}_{m,n}$ and $\mathbf{copy}_m$ are defined using *particular* names; with α-conversion, we shall be justified in choosing these names at will. Throughout this paper we shall adopt the convention that all names appearing in a vector within round brackets are distinct. We also assume that all terms and expressions used are well formed, and when they occur in definitions or equations, those occurring on each side have identical arities.

The equational theory for action calculi consists of a set of equations upon terms generated by the *action structure* axioms and the *control* axioms. The action structure axioms, introduced in [9], state that an action calculus is a strict monoidal category whose objects are given by arities and whose morphisms are defined by terms, with an endofunctor given by the $\mathbf{ab}_x$ operator.

Definition 2.3 **[The theory AC]** The *equational theory* AC is the set of equations upon terms generated by the following axioms:

(i) the action structure axioms

$$
\begin{array}{ll}
A1: & s \cdot \mathbf{id} = s = \mathbf{id} \cdot s \\
A2: & s \otimes \mathbf{id}_\epsilon = s = \mathbf{id}_\epsilon \otimes s \\
A3: & \mathbf{id} \otimes \mathbf{id} = \mathbf{id} \\
A4: & s \cdot (t \cdot u) = (s \cdot t) \cdot u \\
A5: & s \otimes (t \otimes u) = (s \otimes t) \otimes u \\
A6: & (s \cdot t) \otimes (u \cdot v) = (s \otimes u) \cdot (t \otimes v) \\
A7: & \mathbf{ab}_x(\mathbf{id}) = \mathbf{id} \\
A8: & \mathbf{ab}_x(s \cdot t) = \mathbf{ab}_x(s) \cdot \mathbf{ab}_x(t)
\end{array}
$$

(ii) the concrete axioms

$$
\begin{array}{ll}
\gamma : (x)t = \boldsymbol{\omega} \otimes t & (x \notin fn(t)) \\
\delta : (x)(\langle x \rangle \otimes \mathbf{id}_m) = \mathbf{id}_{p \otimes m} & (x : p) \\
\zeta : \mathbf{p}_{k,m} \cdot (t \otimes s) = (s \otimes t) \cdot \mathbf{p}_{l,n} & (s : k \to l,\ t : m \to n) \\
\sigma : (\langle y \rangle \otimes \mathbf{id}_m) \cdot (x)t = t\{y/x\} & (t : m \to n)
\end{array}
$$

For a given set of controls $\mathcal{K}$, we write $s = t \in \mathrm{AC}$ if $s, t \in \mathrm{T}(\mathcal{K})$ and $s = t$ in the equational theory AC. We say that $s = t$ is an *axiom* of AC if it is an instance of the action structure axioms or the concrete axioms.

Lemma 2.4 [Basic lemmas] The following are provable in AC if $x \notin fn(t)$:

(i) $(x)(s \cdot t) = (x)s \cdot t$
(ii) $(x)(s \otimes t) = (x)s \otimes t$
(iii) $(x)(t \otimes s) = t \otimes (x)s,\ t : \varepsilon \to m$
(α) $(y)t = (x)t\{x/y\}$
(iv) $\mathbf{ab}_x(t) = (x)(\langle x \rangle \otimes t)$
(v) $\mathbf{p}_{m,n} \cdot \mathbf{p}_{n,m} = \mathbf{id}$
(vi) $\mathbf{copy}_m \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id}) = \mathbf{id}$
(vii) $\mathbf{copy}_m \cdot (\mathbf{copy}_m \otimes \mathbf{id}) = \mathbf{copy}_m \cdot (\mathbf{id} \otimes \mathbf{copy}_m)$

(viii) $\mathbf{copy}_m \cdot \mathbf{p}_{m,m} = \mathbf{copy}_m$. $\square$

Remark 2.5 Mainly for historical reasons, we have chosen to consider the operator $\mathbf{ab}_x$ as primitive, and define the operator (x) in terms of $\mathbf{ab}_x$. An alternative approach would be to treat (x) as primitive, and let $\mathbf{ab}_x$ be defined by $\mathbf{ab}_x(t) \stackrel{\text{def}}{=} (x)(\langle x\rangle \otimes t)$.

The *action calculus* $\mathrm{AC}(\mathcal{K})$ is defined to be the quotient $\mathrm{T}(\mathcal{K})/\mathrm{AC}$, together with a reaction relation defined below. We view the quotient $\mathrm{T}(\mathcal{K})/\mathrm{AC}$ as the *static* part of the action calculus, and view the reaction relation as the *dynamic* part. The equivalence classes of $\mathrm{T}(\mathcal{K})/\mathrm{AC}$ are called the *actions*. Typically, actions will be denoted by a, b, c. We let $t \in a$ denote that term t is in the equivalence class a.

Definition 2.6 [Dynamics] Associated with each set of controls $\mathcal{K}$ is a set $\mathcal{R}$ of *reaction rules*, each of the form

$$s[\vec{\alpha}] \searrow t[\vec{\alpha}],$$

where $\vec{\alpha}$ are metavariables for terms, and s and t are terms formed from these metavariables and the operations given in definition 2.1, such that $\mathit{fn}(t) \subseteq \mathit{fn}(s)$ and the arities of $s[\vec{\alpha}]$ and $t[\vec{\alpha}]$ are the same. An *instance* of the rule is obtained by replacing the metavariables $\vec{\alpha}$ by terms $\vec{u}$. The *reaction relation* $\searrow$, generated by the set $\mathcal{R}$ of reaction rules, is the smallest reflexive, transitive relation containing every instance of the reaction rules and closed under tensor, composition, abstraction and equality. [3]

Definition 2.7 The action calculus $\mathrm{AC}(\mathcal{K})$ is the quotient $\mathrm{T}(\mathcal{K})/\mathrm{AC}$, together with a reaction relation $\searrow$ given in definition 2.6.

We give two examples of action calculi: $\mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$ which corresponds to part of Milner's π-calculus [8], and $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$ which corresponds to the λ-calculus. In section 5, we use these examples to explore the correspondence between the dynamics of action calculi and their corresponding closed action calculi.

Example 2.8 In this example, we look at the action calculus PIC, discussed in [10], which corresponds to a key fragment of the π-calculus. It is easy to adapt our ideas to the various extensions of PIC studied in [10]. The action calculus $\mathrm{PIC} = \mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$ is based on the underlying monoid $(\mathbf{N}, +, 0)$, given by the set of natural numbers $\mathbf{N}$ with addition for the monoidal tensor and 0 for the unit. It has the arity rules

$$\boldsymbol{\nu} : 0 \to 1 \qquad \mathbf{out} : 1 \otimes m \to 0 \qquad \frac{t : m \to n}{\mathbf{box}(t) : 1 \to n}$$

and the reaction rule

$$(\langle x\rangle \otimes \mathbf{id}_m) \cdot \mathbf{out} \otimes \langle x\rangle \cdot \mathbf{box}(t) \searrow t.$$

[3] Certain other constraints may be placed upon the reaction relations, but they are not necessary for this paper. For example, the requirement that $\mathbf{id} \searrow s$ implies $\mathbf{id} = s$ is usually added. The aim is for the dynamics to capture the behaviour of the controls. More investigation of the dynamics is required in order to capture this intuition.

The control $\boldsymbol{\nu}$ has no associated reaction rule, and is used to restrict access to a name. Sometimes the presentation of the π-calculus as an action calculus also uses the control $\mathbf{in} : 1 \to m$, and the reaction rule $(\langle x\rangle \otimes \mathbf{id}_m)\cdot \mathbf{out} \otimes \langle x\rangle \cdot \mathbf{in} \searrow \mathbf{id}_m$. We choose not to include the control $\mathbf{in}$, since it can be mimicked by the term $\mathbf{box}(\mathbf{id}_m)$. See [10] for more discussion regarding the presentation of the π-calculus as an action calculus.

Example 2.9 Let $M^{\Rightarrow} = (M, \otimes, \Rightarrow, \varepsilon)$ denote a monoid $(M, \otimes, \varepsilon)$ with the binary operator $\Rightarrow$ on arities freely added. If $(M, \otimes, \varepsilon)$ has the set of primes P, then $M^{\Rightarrow}$ has the set of primes $P \cup \{m \Rightarrow n : m, n \in M^{\Rightarrow}\}$. We define an action calculus $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$ with arity rules

$$\frac{t : m \to n}{\ulcorner t \urcorner : \varepsilon \to m \Rightarrow n} \qquad\qquad \mathbf{ap} : (m \Rightarrow n) \otimes n \to n$$

and the reaction rules

$$\sigma' : \qquad (\ulcorner t \urcorner \otimes \mathbf{id}) \cdot (x)s \searrow s\{t/x\}, \qquad \text{for } x : m \Rightarrow n$$

$$\beta : \qquad (\ulcorner t \urcorner \otimes \mathbf{id}) \cdot \mathbf{ap} \searrow t$$

where $s\{t/x\}$ denotes the substitution of any occurrence of $\langle x\rangle$ by the term $\ulcorner t \urcorner$. This action calculus is introduced in [11]. It intuitively corresponds to the simply-typed λ-calculus, where the σ' reaction rule corresponds to explicit substitution and the β reaction rule corresponds to β-reduction. Further work is required in order to make this correspondence precise.

2.1 Contextual Action Calculi

The contextual action calculi provide a type-theoretic presentation of action calculi, which give a local account of names using contexts. The connection between action calculi and their corresponding closed action calculi is given by first establishing the link between action calculi and contextual action calculi, and then showing an exact correspondence between contextual action calculi and closed action calculi.

Definition 2.10 The *equational theory with names*, denoted by AC_n, is defined by the following rules, where $\{\vec{x}\}$ denotes a set of distinct names and the arity information is omitted since it is apparent [4]:

$$\{\vec{x}\} \vdash s = t, \qquad\qquad s = t \text{ an axiom of AC}, \mathit{fn}(s) \cup \mathit{fn}(t) \subseteq \{\vec{x}\}$$

$$\{\vec{x}\} \vdash s = s, \qquad\qquad \mathit{fn}(s) \subseteq \{\vec{x}\}$$

$$\frac{\{\vec{x}\} \vdash s = t}{\{\vec{x}\} \vdash t = s}$$

$$\frac{\{\vec{x}\} \vdash s = t \quad \{\vec{x}\} \vdash t = u}{\{\vec{x}\} \vdash s = u}$$

[4] We use a rule with two conclusions as shorthand for two rules with the same premises and one conclusion each.

$$\frac{\{\vec{x},y\} \vdash s = t}{\{\vec{x}\} \vdash (y)s = (y)t} \qquad y \notin \{\vec{x}\}$$

$$\frac{\{\vec{x}\} \vdash s = t}{\{\vec{x}\} \vdash u \cdot s = u \cdot t \qquad \{\vec{x}\} \vdash s \cdot u = t \cdot u} \qquad fn(u) \subseteq \{\vec{x}\}$$

$$\frac{\{\vec{x}\} \vdash s = t}{\{\vec{x}\} \vdash u \otimes s = u \otimes t \qquad \{\vec{x}\} \vdash s \otimes u = t \otimes u} \qquad fn(u) \subseteq \{\vec{x}\}$$

$$*\frac{\{\vec{x}\} \vdash s_i = t_i, \qquad i = 1, \ldots, r}{\{\vec{x}\} \vdash K(s_1, \ldots, s_r) = K(t_1, \ldots, t_r)} \qquad \chi$$

For a given set $\mathcal{K}$ of controls, we write $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$ if $s, t \in \mathrm{T}(\mathcal{K})$ and $\{\vec{x}\} \vdash s = t$ can be proved using the above rules.

Proposition 2.11

(i) $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$ implies $fn(s) \cup fn(t) \subseteq \{\vec{x}\}$.

(ii) $\{\vec{x}, y\} \vdash s = t \in \mathrm{AC}_n$ and $z \notin \{\vec{x}\}$ imply $\{\vec{x}, z\} \vdash s\{z/y\} = t\{z/y\} \in \mathrm{AC}_n$.

(iii) $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$ and $y \notin \{\vec{x}\}$ imply $\{\vec{x}, y\} \vdash s = t \in \mathrm{AC}_n$.

(iv) $\{\vec{x}, y\} \vdash s = t \in \mathrm{AC}_n$ and $y \notin fn(s,t)$ imply $\{\vec{x}\} \vdash s = t \in \mathrm{AC}_n$.

Proof. The proofs of parts (i) to (iii) are easy. The proof of part (iv) is less straightforward. It relies on the connection between $\mathrm{AC}_n(\mathcal{K})$ and the alternative presentation of actions using the molecular forms. See [2] for a detailed proof. □

Proposition 2.12 $s = t \in \mathrm{AC}$ if and only if $fn(s,t) \vdash s = t \in \mathrm{AC}_n$. □

The *contextual action calculus* $\mathrm{AC}_n(\mathcal{K})$ is defined to be the quotient $\mathrm{T}(\mathcal{K})/\mathrm{AC}_n$, together with a reaction relation, defined as follows.

Definition 2.13 Let $\mathcal{R}$ be a set of reaction rules as described in definition 2.6. The *reaction relation* $\searrow$ generated by $\mathcal{R}$, is the smallest relation given by the following rules, where we assume that $\{\vec{x}\}$ denotes a distinct set of names:

$$\{\vec{x}\} \vdash s \searrow s \qquad fn(s) \subseteq \{\vec{x}\}$$

$$\{\vec{x}\} \vdash s \searrow t \qquad s \searrow t \text{ is an instance of } \mathcal{R},\ fn(s,t) \subseteq \{\vec{x}\}$$

$$\frac{\{\vec{x}\} \vdash s \searrow t \quad \{\vec{x}\} \vdash t \searrow u}{\{\vec{x}\} \vdash s \searrow u}$$

$$\frac{\{\vec{x}\} \vdash s \searrow t}{\{\vec{x}\} \vdash u \cdot s \searrow u \cdot t \qquad \{\vec{x}\} \vdash s \cdot u \searrow t \cdot u} \qquad fn(u) \subseteq \{\vec{x}\}$$

$$\frac{\{\vec{x}\} \vdash s \searrow t}{\{\vec{x}\} \vdash u \otimes s \searrow u \otimes t \qquad \{\vec{x}\} \vdash s \otimes u \searrow t \otimes u} \qquad fn(u) \subseteq \{\vec{x}\}$$

$$\frac{\{\vec{x}, y\} \vdash s \searrow t}{\{\vec{x}\} \vdash (y)s \searrow (y)t} \qquad y \notin \{\vec{x}\}$$

$$\frac{\{\vec{x}\} \vdash s = u \quad \{\vec{x}\} \vdash s \searrow t \quad \{\vec{x}\} \vdash t = v}{\{\vec{x}\} \vdash u \searrow v}$$

Proposition 2.14

(i) $\{\vec{x}\} \vdash s \searrow t$ implies $fn(s,t) \subseteq \{\vec{x}\}$.
(ii) $\{\vec{x}, y\} \vdash s \searrow t$ and $z \notin \{\vec{x}\}$ implies $\{\vec{x}, z\} \vdash s\{z/y\} \searrow t\{z/y\}$.
(iii) $\{\vec{x}\} \vdash s \searrow t$ and $y \notin \{\vec{x}\}$ implies $\{\vec{x}, y\} \vdash s \searrow t$. □

Proposition 2.15 If the reaction relations for $\mathrm{AC}(\mathcal{K})$ and $\mathrm{AC}_n(\mathcal{K})$ are generated by the same set of reaction rules $\mathcal{R}$, then $s \searrow t \in \mathrm{AC}(\mathcal{K})$ if and only if $\{\vec{x}\} \vdash s \searrow t \in \mathrm{AC}_n(\mathcal{K})$, for some set $\{\vec{x}\}$ of distinct names. □

Remark 2.16 It looks as if the stronger result that $s \searrow t \in \mathrm{AC}$ implies $fn(s,t) \vdash s \searrow t \in \mathrm{AC}_n$ does not always hold. For example, consider the reaction rule $\langle x \rangle \cdot \boldsymbol{\omega} \otimes K \searrow \mathbf{id}$. We have $\{x\} \vdash K \searrow \mathbf{id}$, but $\emptyset \vdash K \searrow \mathbf{id}$ does not seem to hold. There is in fact a simple condition on reaction rules under which the stronger result holds. This condition is satisfied by all the examples studied by Milner in [10] and [11], including the action calculi $\mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$ and $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$ given in examples 2.8 and 2.9 respectively. We do not explore this condition, since the dynamics for action calculi are not fully understood yet.

3 Closed Action Calculi

Using an analogous approach to the definition of action calculi, we define a closed action calculus as a quotient of a term algebra constructed from an underlying monoid $(M, \otimes, \epsilon)$. In particular, given an action calculus $\mathrm{AC}(\mathcal{K})$, we distinguish the corresponding closed action calculus $\mathrm{CAC}(\mathcal{K}')$. Section 4 contains the translations and results which explain the correspondence between $\mathrm{AC}(\mathcal{K})$ and $\mathrm{CAC}(\mathcal{K}')$.

A closed action calculus $\mathrm{CAC}(\mathcal{K})$ possesses a set $\mathcal{K}$ of controls, each equipped with an arity rule. Each $\mathrm{CAC}(\mathcal{K})$ is determined by its controls, together with a set of reaction rules which define its dynamics. The closed terms have the form $t : m \to n$, for $m, n \in (M, \otimes, \epsilon)$, where t is constructed from the basic operators $\mathbf{id}_m, \boldsymbol{\omega}_m, \Delta_m, \mathbf{i}_{m,n}, \cdot, \otimes$ and the controls $K \in \mathcal{K}$. The operators Δ_m and $\mathbf{i}_{m,n}$ correspond to the copying and permutation operators given in definition 2.2, as is apparent from the axioms accompanying these operators. This correspondence is expressed formally in section 4. The other operators are self-explanatory.

Definition 3.1 **[Closed Terms]** The *set of closed terms* over $\mathcal{K}$, denoted by $\mathrm{CT}(\mathcal{K})$, is generated by the following rules:

$$\mathbf{id}_m : m \to m$$

$$\frac{s : k \to l \qquad t : l \to m}{s \cdot t : k \to m}$$

$$\frac{s : k \to m \qquad t : l \to n}{s \otimes t : k \otimes l \to m \otimes n}$$

$$\Delta_m : m \to m \otimes m$$

$$\mathbf{i}_{m,n} : m \otimes n \to n \otimes m$$

$$\boldsymbol{\omega}_m : m \to \epsilon$$

$$* \frac{t_1 : m_1 \to n_1 \quad \ldots \quad t_r : m_r \to n_r}{K(t_1, \ldots, t_r) : m \to n} \chi$$

where each control $K \in \mathcal{K}$ is accompanied by an arity rule $*$, such that side-condition χ may constrict the value of the integer r and the arities m_i, n_i, m and n. We shall omit the arity subscripts on the basic operators when they are apparent.

Definition 3.2 [**The Theory CAC**] The equational theory CAC is the set of equations upon terms generated by the action structure axioms A1–A6 from section 2, and the following:

$$
\begin{aligned}
B1 :&\quad \Delta_m \cdot (\boldsymbol{\omega}_m \otimes \mathbf{id}) = \mathbf{id} \\
B2 :&\quad \Delta_m \cdot \mathbf{i}_{m,m} = \Delta_m \\
B3 :&\quad \mathbf{i}_{k,m} \cdot (s \otimes t) = (t \otimes s) \cdot \mathbf{i}_{l,n} \\
B4 :&\quad \mathbf{i}_{m,n} \cdot \mathbf{i}_{n,m} = \mathbf{id} \\
B5 :&\quad \mathbf{i}_{m \otimes n,k} = (\mathbf{id} \otimes \mathbf{i}_{n,k}) \cdot (\mathbf{i}_{m,n} \otimes \mathbf{id}) \\
B6 :&\quad \boldsymbol{\omega}_{m \otimes n} = \boldsymbol{\omega}_m \otimes \boldsymbol{\omega}_n \\
B7 :&\quad \Delta_{m \otimes n} = (\Delta_m \otimes \Delta_n) \cdot (\mathbf{id} \otimes \mathbf{i}_{m,n} \otimes \mathbf{id}) \\
B8 :&\quad \Delta_m \cdot (\Delta_m \otimes \mathbf{id}) = \Delta_m \cdot (\mathbf{id} \otimes \Delta_m)
\end{aligned}
$$

For a given set of controls $\mathcal{K}$, we write $s = t \in \mathrm{CAC}$ if $s, t \in \mathrm{CT}(\mathcal{K})$ and $s = t$ is in the equational theory CAC.

Remark 3.3 We have chosen to define $\mathbf{id}_m$, $\boldsymbol{\omega}_m$, Δ_m and $\mathbf{i}_{m,n}$ for arbitrary arities and include the axioms B5–B7. Since arities can be uniquely factorized into primes, an alternative approach is to restrict the definitions to prime arities, remove B5–B7 and define the composite cases in terms of the prime cases and the other operators. This alternative approach is used in the definition of action calculi, since names are forced to have prime arity.

The *closed* action calculus $\mathrm{CAC}(\mathcal{K})$ is defined to be the quotient $\mathrm{CT}(\mathcal{K})/$ CAC, together with a reaction relation which we define below. We call the elements of $\mathrm{CT}(\mathcal{K})/\mathrm{CAC}$ the *closed actions*. Typically, closed actions will be denoted by a, b, c; it should be clear from the context when a denotes an action or a closed action. We also let $t \in a$ denote that closed term t inhabits the equivalence class denoted by a.

Definition 3.4 [**Dynamics**] Let $\mathcal{R}$ be a set of reaction rules for closed terms, defined in an analogous way to the rules given in definition 2.6. The *reaction relation* $\searrow$ generated by $\mathcal{R}$ is the smallest reflexive, transitive relation containing every instance of the reaction rules and closed under tensor, composition and equality.

Definition 3.5 The closed action calculus CAC($\mathcal{K}$) is given by the quotient CT($\mathcal{K}$)/CAC, together with a reaction relation $\searrow$ given in definition 3.4.

Given an action calculus AC($\mathcal{K}$), we distinguish the corresponding closed action calculus CAC($\mathcal{K}'$), where the set of controls $\mathcal{K}'$ is uniquely determined by the set $\mathcal{K}$. The free names of an action calculus provide an interface between the terms inside controls and the rest of the term. For example, using the action calculus AC($\boldsymbol{\nu}$, **out**, **box**), given in example 2.8, we have

$$(\langle z,z\rangle \otimes \mathbf{id}) \cdot (x,y)\mathbf{box}(\langle x,y\rangle) = \mathbf{box}(\langle z,z\rangle).$$

In order to mimic this behaviour in the closed world, we declare, for each $K \in \mathcal{K}$, the controls $K_n \in \mathcal{K}'$ for every $n \in (M, \otimes, \mathbf{id})$. The purpose of the index n is to record the fact that terms inside the control K_n have been closed with respect to some sequence of names $\vec{x}$, where $|\vec{x}| = n$. For example, if we close the term $\mathbf{box}(\langle x,y\rangle)$ using sequence $[x:p, y:q]$ we obtain the closed term $\mathbf{box}_{p\otimes q}(\mathbf{id})$. If we close the same term using sequence $[y:q, x:p]$, we obtain the closed term $\mathbf{box}_{q\otimes p}(\mathbf{i}_{q,p})$. Intuitively, these two closed terms should be connected since they have come from the same term $\mathbf{box}(\langle x,y\rangle)$. This intuition is captured by adding extra equalities to link controls with related indexing. For example, the controls $\mathbf{box}_{p\otimes q}$ and $\mathbf{box}_{q\otimes p}$ are connected by the equality

$$\mathbf{i}_{p,q} \cdot \mathbf{box}_{q\otimes p}((\mathbf{i}_{q,p} \otimes \mathbf{id}) \cdot t) = \mathbf{box}_{p\otimes q}(t),$$

which results in $\mathbf{box}_{p\otimes q}(\mathbf{id})$ and $(\mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot \mathbf{box}_{q\otimes p}(\mathbf{i}_{q,p})$ being equal. Using these extra equalities on the indexed controls, we obtain a tight correspondence between the equational theories AC_n and CAC.

Definition 3.6 The *closed action calculus for* AC($\mathcal{K}$), denoted by CAC($\mathcal{K}'$), is a closed action calculus with the same underlying monoid as AC($\mathcal{K}$), such that

$$\mathcal{K}' = \{K_n : n \in (M, \otimes, \epsilon) \text{ and } K \in \mathcal{K}\}$$

and if the arity rule accompanying K is

$$\frac{t_1 : m_1 \to n_1 \quad \ldots \quad t_r : m_r \to n_r}{K(t_1, \ldots, t_r) : k \to l}\chi$$

then the arity rule accompanying K_p for each $p \in (M, \otimes, \epsilon)$ is

$$\frac{t_1 : p \otimes m_1 \to n_1 \quad \ldots \quad t_r : p \otimes m_r \to n_r}{K_p(t_1, \ldots, t_r) : p \otimes k \to l}\chi$$

We also require the following *control axioms*:

1. $K_{q\otimes p}(\boldsymbol{\omega}_q \otimes t_1, \ldots, \boldsymbol{\omega}_q \otimes t_r) = \boldsymbol{\omega}_q \otimes K_p(t_1, \ldots, t_r)$

2. $K_{k\otimes p\otimes q}((\mathbf{id} \otimes \mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot t_1, \ldots, (\mathbf{id} \otimes \mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot t_r) = (\mathbf{id} \otimes \mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot K_{k\otimes q\otimes p}(t_1, \ldots, t_r)$

3. $K_{k\otimes p}((\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot t_1, \ldots, (\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot t_r) = (\mathbf{id}\otimes\Delta_p\otimes\mathbf{id})\cdot K_{k\otimes p\otimes p}(t_1, \ldots, t_r)$

The above control axioms are necessary to prove lemma 4.3. We intend that, given the reaction relation accompanying $\mathrm{AC}(\mathcal{K})$, there is a corresponding relation in $\mathrm{CAC}(\mathcal{K}')$. The dynamics for action calculi have not yet been fully explored, and so a general account connecting the dynamics of action calculi and their corresponding closed action calculi is not yet possible. We briefly explore the connection between the dynamics of $\mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$ and $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$, and their corresponding closed action calculi in section 5.

4 Translations

This section contains the formal justification for introducing the closed action calculi. We define the *closure functions* $[\![\_]\!]_{\vec{x}} : \mathrm{T}_{\{\vec{x}\}}(\mathcal{K}) \to \mathrm{CT}(\mathcal{K}')$ and the function $\langle\_\rangle : \mathrm{CT}(\mathcal{K}') \to T_{\emptyset}(\mathcal{K})$, which preserve the equalities given by AC and CAC. These functions provide a close correspondence between AC and CAC, in the sense that $\langle [\![t]\!]_{\vec{x}} \rangle = (\vec{x})t \in \mathrm{AC}$, whenever $fn(t) \subseteq \{\vec{x}\}$, and $[\![\langle t \rangle]\!]_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|} \otimes t \in \mathrm{CAC}$. The proofs are given in [3].

4.1 Action Calculi to Closed Action Calculi

As the name suggests, we intuitively regard the closure function $[\![\_]\!]_{\vec{x}}$ as closing up the terms in $\mathrm{T}_{\{\vec{x}\}}(\mathcal{K})$ using the sequence of variables $\vec{x}$. Recall from the previous section that we use the arity indexing of controls in $\mathcal{K}'$ to record the information that terms inside controls have been closed using a sequence of variables of the appropriate arities.

Definition 4.1 The *closure functions* $[\![\_]\!]_{\vec{x}} : T_{\{\vec{x}\}}(\mathcal{K}) \to \mathrm{CT}(\mathcal{K}')$, for each distinct list of names $\vec{x} = [x_1 : p_1, \ldots, x_r : p_r]$, are defined inductively on the structure of terms in $\mathrm{T}_{\{\vec{x}\}}(K)$ as follows:

$$
\begin{aligned}
[\![\mathbf{id}]\!]_{\vec{x}} &= \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{id} \\
[\![s \cdot t]\!]_{\vec{x}} &= (\Delta_{|\vec{x}|} \otimes \mathbf{id}) \cdot (\mathbf{id}_{|\vec{x}|} \otimes [\![s]\!]_{\vec{x}}) \cdot [\![t]\!]_{\vec{x}} \\
[\![s \otimes t]\!]_{\vec{x}} &= (\Delta_{\vec{x}} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{|\vec{x}|,k} \otimes \mathbf{id}) \cdot ([\![s]\!]_{\vec{x}} \otimes [\![t]\!]_{\vec{x}}) \\
[\![(x)t]\!]_{\vec{x}} &= [\![t\{y/x\}]\!]_{\vec{x},y}, \quad y \notin \{\vec{x}\} \\
[\![\langle x \rangle]\!]_{\vec{x}} &= \boldsymbol{\omega}_{p_i \otimes \ldots \otimes p_{i-1}} \otimes \mathbf{id}_{p_i} \otimes \boldsymbol{\omega}_{p_{i+1} \otimes \ldots \otimes p_r}, \quad x = x_i \\
[\![\boldsymbol{\omega}_p]\!]_{\vec{x}} &= \boldsymbol{\omega}_{|\vec{x}|} \otimes \boldsymbol{\omega}_p \\
[\![K(t_1, \ldots, t_r)]\!]_{\vec{x}} &= K_{|\vec{x}|}([\![t_1]\!]_{\vec{x}}, \ldots, [\![t_r]\!]_{\vec{x}})
\end{aligned}
$$

Whenever we write $[\![\_]\!]_{\vec{x}}$, we assume that $\vec{x}$ is a list of distinct names. We shall often wish to distinguish a particular name in such a list. We therefore write

$\vec{x}, y, \vec{z}$ to denote a sequence of distinct names with the name y distinguished. We intuitively regard the operators Δ_m and $\mathbf{i}_{m,n}$ in CAC as having the same role as the operators $\mathbf{copy}_m$ and $\mathbf{p}_{m,n}$ in AC. It is not difficult to show that $[\![\mathbf{copy}_m]\!]_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|} \otimes \Delta_m \in \mathrm{CAC}$ and $[\![\mathbf{p}_{m,n}]\!]_{\vec{x}} = \boldsymbol{\omega}_{|\vec{x}|} \otimes \mathbf{i}_{m,n} \in \mathrm{CAC}$, which partly justifies this intuition.

Notice that $[\![(x)t]\!]_{\vec{x}}$ is defined using a chosen $y \notin \{\vec{x}\}$. The next lemma shows that this choice of y is not important.

Lemma 4.2 $\quad [\![t]\!]_{\vec{x},u,\vec{y}} = [\![t\{v/u\}]\!]_{\vec{x},v,\vec{y}} \in \mathrm{CAC}$, if $u : p$ and $v : p$. $\square$

The following three lemmas illustrate the connection between the closure functions $[\![\_]\!]_{\vec{x}}$ and $[\![\_]\!]_{\vec{y}}$, when $\{\vec{x}\} \subseteq \{\vec{y}\}$. They are proved by induction on the structure of term t. In each proof, the interesting case is when t has the form $K(t_1, \ldots, t_r)$, since this case shows that the proofs rely directly on the control axioms introduced in definition 3.6.

Lemma 4.3

(i) $[\![t]\!]_{y,\vec{x}} = \boldsymbol{\omega}_p \otimes [\![t]\!]_{\vec{x}} \in \mathrm{CAC}$, when $y : p \notin fn(t)$.

(ii) $[\![t]\!]_{\vec{x},\vec{y},\vec{z}} = (\mathbf{id} \otimes \mathbf{i}_{|\vec{y}|,|\vec{z}|} \otimes \mathbf{id}) \cdot [\![t]\!]_{\vec{x},\vec{z},\vec{y}} \in \mathrm{CAC}$.

(iii) $[\![t\{u/v\}]\!]_{\vec{x},u,\vec{y}} = (\mathbf{id} \otimes \Delta_p \otimes \mathbf{id} \otimes \mathbf{id}) \cdot [\![t]\!]_{\vec{x},u,v,\vec{y}} \in \mathrm{CAC}$, for $u : p$ and $v : p$. $\square$

Using lemma 4.3, we are able to prove that the closure functions preserve the equalities given by AC_n. Lemma 4.3 is used to show that the axioms γ and σ are preserved under the translation.

Theorem 4.4 $\quad \{\vec{x}\} \vdash s = t \in \mathrm{AC}_n(\mathcal{K})$ implies $[\![s]\!]_{\vec{x}} = [\![t]\!]_{\vec{x}} \in \mathrm{CAC}$. $\square$

From the above theorem and proposition 2.12, we infer that $s = t \in \mathrm{AC}$ implies $[\![s]\!]_{\mathrm{fn}(s,t)} = [\![t]\!]_{\mathrm{fn}(s,t)} \in \mathrm{CAC}$.

4.2 Closed Action Calculi to Action Calculi

In this section, we define the translation $\langle\_\rangle : \mathrm{CT}(\mathcal{K}') \to \mathrm{T}_\emptyset(\mathcal{K})$ which preserves the equalities given by CAC and the control axioms. This translation, together with the closure functions defined in the previous section, yields a tight correspondence between the static parts of $\mathrm{AC}(\mathcal{K})$ and $\mathrm{CAC}(\mathcal{K}')$.

Using the intuition that the operators Δ_m and $\mathbf{i}_{m,n}$ in CAC play essentially the same role as the operators $\mathbf{copy}_m$ and $\mathbf{p}_{m,n}$ in AC, we view the translation $\langle\_\rangle$ as the identity function in all cases, except the control case. Recall that the indexing in $\mathcal{K}'$ is used to record the information that the terms inside the controls have been closed using a sequence of variables of the appropriate arity. We use this information during translation in an essential way to incorporate free variables inside the controls.

Definition 4.5 The translation $\langle\_\rangle : \mathrm{CT}(\mathcal{K}') \to \mathrm{T}_\emptyset(\mathcal{K})$ is defined inductively on the structure of closed terms as follows:

$$\langle \mathbf{id} \rangle = \mathbf{id}$$

$$\langle s \cdot t \rangle = \langle s \rangle \cdot \langle t \rangle$$

$$\langle s \otimes t \rangle = \langle s \rangle \otimes \langle t \rangle$$

$$\begin{aligned}
\langle \Delta_m \rangle &= \mathbf{copy}_m \\
\langle \mathbf{i}_{m,n} \rangle &= \mathbf{p}_{m,n} \\
\langle \omega \rangle &= \omega \\
\langle K_m(t_1,\ldots,t_r) \rangle &= (\vec{x})K(((\langle \vec{x} \rangle \otimes \mathbf{id}) \cdot \langle s \rangle, \ldots, ((\langle \vec{x} \rangle \otimes \mathbf{id}) \cdot \langle t_r \rangle), \\
&\qquad \text{where } m = p_1 \otimes \ldots \otimes p_s \text{ and } \vec{x} \text{ is } [x_1 : p_1, \ldots, x : s : p_s].
\end{aligned}$$

Theorem 4.6 $s = t \in \mathrm{CAC}$ implies $\langle s \rangle =_{\emptyset} \langle t \rangle \in \mathrm{AC}_n$ □

From the above theorem and proposition 2.12, we infer that $s = t \in \mathrm{CAC}$ implies $\langle s \rangle = \langle t \rangle \in \mathrm{AC}$. There is a tight correspondence between equalities in AC and equalities in CAC, as the following theorem states.

Theorem 4.7

(i) $\langle [\![t]\!]_{\vec{x}} \rangle = (\vec{x})t \in \mathrm{AC}$, if $fn(t) \subseteq \{\vec{x}\}$.

(ii) $[\![\langle s \rangle]\!]_{\vec{x}} = \omega_{|\vec{x}|} \otimes s \in \mathrm{CAC}$. □

Corollary 4.8

(i) $[\![s_1]\!]_{\vec{x}} = [\![s_2]\!]_{\vec{x}} \in \mathrm{CAC}$ implies $s_1 = s_2 \in \mathrm{AC}$, when $fn(s_1, s_2) \subseteq \{\vec{x}\}$.

(ii) $\langle t_1 \rangle = \langle t_2 \rangle \in \mathrm{AC}_n$ implies $t_1 = t_2 \in \mathrm{CAC}$. □

5 Dynamics

A general account connecting the reaction relations of $\mathrm{AC}(\mathcal{K})$ and $\mathrm{CAC}(\mathcal{K}')$ is beyond the scope of this paper. In this section, we establish the connection between the reaction relations of $\mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$ and $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$, and their corresponding closed action calculi, to illustrate that such a correspondence is possible in these cases. The first example is straightforward, with one reaction rule in the open world corresponding to one reaction rule in the closed world. The second example requires more care, in that the number of reaction rules in the open and closed world are not the same.

The first example is the action calculus $\mathrm{AC}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$, given in example 2.8. The corresponding closed action calculus $\mathrm{CAC}(\boldsymbol{\nu}_m, \mathbf{out}_m, \mathbf{box}_m)$ is given by definition 3.6 and the reaction rule

$$(\Delta_{k\otimes 1} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \mathbf{i}_{k\otimes 1,m}) \cdot (\mathbf{out}_k \otimes \mathbf{box}_k(t)) \searrow (\mathbf{id} \otimes \omega_1 \otimes \mathbf{id}) \cdot t$$

The translations $[\![\_]\!]_{\vec{x}} : \mathrm{T}_{\{\vec{x}\}}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box}) \to \mathrm{CT}(\boldsymbol{\nu}_m, \mathbf{out}_m, \mathbf{box}_m)$ and $\langle \_ \rangle :$ $\mathrm{CT}(\boldsymbol{\nu}_m, \mathbf{out}_m, \mathbf{box}_m) \to \mathrm{T}_{\emptyset}(\boldsymbol{\nu}, \mathbf{out}, \mathbf{box})$ preserve the reaction relations as the following theorem states.

Theorem 5.1

(i) $\{\vec{x}\} \vdash s \searrow t \in \mathrm{AC}_n$ implies $[\![s]\!]_{\vec{x}} \searrow [\![t]\!]_{\vec{x}} \in \mathrm{CAC}$.

(ii) $s \searrow t \in \mathrm{CAC}$ implies $\emptyset \vdash \langle s \rangle \searrow \langle t \rangle \in \mathrm{AC}_n$. □

In fact, we also have the stronger result that $s \searrow t \in \mathrm{AC}$ implies $[\![s]\!]_{fn(s,t)} \searrow [\![t]\!]_{fn(s,t)} \in \mathrm{CAC}$. This stronger result does not hold in general, as mentioned in remark 2.16.

Corollary 5.2

(i) $[\![s]\!]_{\vec{x}} \searrow t \in \mathrm{CAC}$ implies $s \searrow s' \in \mathrm{AC}$ and $[\![s']\!]_{\vec{x}} = t \in \mathrm{CAC}$.

(ii) $\langle s \rangle \searrow t \in \mathrm{AC}$ implies $s \searrow s' \in \mathrm{CAC}$ and $\langle s' \rangle = t \in \mathrm{AC}$. $\square$

The second example is the action calculus $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$, defined in example 2.9. Its corresponding closed action calculus $\mathrm{CAC}(\ulcorner\_\urcorner_m, \mathbf{ap}_m)$ is given by definition 3.6 and the reaction rules

$$
\begin{array}{rcll}
\ulcorner t \urcorner_k \cdot \Delta_{m \Rightarrow n} & \searrow & \Delta_k \cdot (\ulcorner t \urcorner_k \otimes \ulcorner t \urcorner_k) & (1)\\
\ulcorner t \urcorner_k \cdot \omega_{m \Rightarrow n} & \searrow & \omega_k & (2)\\
(\ulcorner t \urcorner_k \otimes \mathbf{id}) \cdot \mathbf{ap}_k & \searrow & t & (3)
\end{array}
$$

The reaction rules 1 and 2 are used to mimic in the closed world the substitution of a term t for a name in the open world. This example is not as straightforward as the previous example, since two reaction rules in the closed world correspond to one reaction rule in the open world. In general, we can have an arbitrary number of reaction rules in the closed world corresponding to one reaction rule in the open world. For example, given the action calculus $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap}, \mathcal{K})$, for an arbitrary control set $\mathcal{K}$, then the corresponding closed action calculus would contain, for each $K \in \mathcal{K}$, a reaction rule

$$
\begin{array}{l}
(\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner b \urcorner_k \otimes \mathbf{id}) \cdot K_{k \otimes m \Rightarrow n}(t_1, \dots, t_r) \searrow \\
K_k((\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner b \urcorner_k \otimes \mathbf{id}) \cdot t_1, \dots, (\Delta_k \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner b \urcorner_k \otimes \mathbf{id}) \cdot t_r)
\end{array}
$$

The following lemma is used to prove the connection between the dynamics of $\mathrm{AC}(\ulcorner\_\urcorner, \mathbf{ap})$ and $\mathrm{CAC}(\ulcorner\_\urcorner_m, \mathbf{ap}_m)$.

Lemma 5.3 $(\Delta_{|\vec{y}|} \otimes \mathbf{id}) \cdot (\mathbf{id} \otimes \ulcorner [\![t]\!]_{\vec{y}} \urcorner_{|\vec{y}|} \otimes \mathbf{id}) \cdot [\![s\{u/x\}]\!]_{\vec{y},u} \searrow [\![s\{t/x\}]\!]_{\vec{y}} \in \mathrm{CAC}$, for $x, u : m \Rightarrow n$. $\square$

Theorem 5.4

(i) $\{\vec{x}\} \vdash s \searrow t \in \mathrm{AC}_n$ implies $[\![s]\!]_{\vec{x}} \searrow [\![t]\!]_{\vec{x}} \in \mathrm{CAC}$.

(ii) $s \searrow t \in \mathrm{CAC}$ implies $\emptyset \vdash \langle s \rangle \searrow \langle t \rangle \in \mathrm{AC}_n$. $\square$

Again, we have the stronger result that $s \searrow t \in \mathrm{AC}$ implies $[\![s]\!]_{fn(\mathrm{s,t})} \searrow [\![t]\!]_{fn(\mathrm{s,t})} \in \mathrm{CAC}$.

6 Adding Reflexion

In this section, we look at the *reflexive* action calculi, introduced by Milner in [13], and show that adding reflexion to the corresponding closed action calculi is straightforward. The full details can be found in [13]. Reflexive calculi are action calculi with extra structure given by a *reflexive* operator $\uparrow_p$, with the accompanying arity rule

$$\frac{t : p \otimes m \to p \otimes n}{\uparrow_p(t) : m \to n}$$

and additional axioms to account for the behaviour of $\uparrow_p$. The closed reflexive action calculi are also constructed using a reflexive operator, with the same accompanying arity rule and a similar set of axioms to describe the operator.

This treatment of adding extra structure in the closed world to mimic the extra structure in the open world is very different to the approach taken if we had declared $\uparrow_p$ as a control. In the closed world, controls should be indexed with closure information; it looks as if additional structural operators need not.

The set of *reflexive terms* over $\mathcal{K}$, denoted by $\mathrm{RT}(\mathcal{K})$, are constructed using the operators of action calculi with their associated arity rules, plus the reflexion operator $\uparrow_p$ with its arity rule given above. The equational theory RAC is given by the set of equations upon reflexive terms generated by the action structure axioms, the axioms for AC and the following additional axioms which account for the behaviour of $\uparrow_p$:

$$\begin{aligned}
\rho 1 :&\quad \mathbf{id} = \uparrow_p \mathbf{i}_{p,p}\\
\rho 2 :&\quad \uparrow_p t \otimes \mathbf{id} = \uparrow_p (t \otimes \mathbf{id})\\
\rho 3 :&\quad \uparrow_p s \cdot t = \uparrow_p (s \cdot (\mathbf{id}_p \otimes t))\\
\rho 4 :&\quad s \cdot \uparrow_p t = \uparrow_p ((\mathbf{id}_p \otimes s) \cdot t)\\
\rho 5 :&\quad \uparrow_q \uparrow_p t = \uparrow_p \uparrow_q ((\mathbf{i}_{q,p} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{p,q} \otimes \mathbf{id}))
\end{aligned}$$

Remark 6.1 In the original definition of reflexive action calculi [13], we also have the axiom

$$(x) \uparrow_p t = \uparrow_p ((\mathbf{i}_{p,q} \otimes \mathbf{id}) \cdot (x) t), \qquad (x : q)$$

Hasegawa [4] has recently observed that this axiom follows from the action calculi axioms and $\rho 1$–$\rho 5$.

For a given set of controls $\mathcal{K}$, we write $s = t \in \mathrm{RAC}$ if $s, t \in \mathrm{RT}(\mathcal{K})$ and $s = t$ in the equational theory RAC. We call $\rho 1$–$\rho 5$ the reflexion axioms. It is natural to define the iterated reflexion as follows:

$$\uparrow_p t \stackrel{\mathrm{def}}{=} \uparrow_{p_r} \dots \uparrow_{p_1} t, \quad m = p_1 \otimes \dots \otimes p_r$$

Lemma 6.2 [Basic Lemmas] The following are provable in RAC.

(i) $s \cdot t = \uparrow_m (\mathbf{i}_{m,k} \cdot (s \otimes t)), \qquad s : k \to m, t : m \to n$
(ii) $(s \otimes \mathbf{id}_k) \cdot t = \uparrow_m (s \otimes t), \qquad s : \varepsilon \to m, t : m \otimes k \to n$
(iii) $\uparrow_m s \otimes t = \uparrow_m (s \otimes t)$
(iv) $\uparrow_m s \cdot t = \uparrow_m (s \cdot (\mathbf{id}_m \otimes t))$
(v) $s \cdot \uparrow_m t = \uparrow_m ((\mathbf{id}_m \otimes s) \cdot t)$
(vi) $(\vec{x}) \uparrow_m t = \uparrow_m ((\mathbf{i}_{m,n} \otimes \mathbf{id}) \cdot (\vec{x}) t) \qquad (\vec{x} : n)$
(vii) $\uparrow_n \uparrow_m t = \uparrow_m \uparrow_n ((\mathbf{i}_{n,m} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{m,n} \otimes \mathbf{id}))$ □

The reflexive equational theory with names, denoted by RAC_n, is defined using the rules given in the definition of AC_n (definition 2.10), where in this case the axioms include the appropriate reflexion axioms given above.

The *closed reflexive terms over* $\mathcal{K}$, denoted by $\mathrm{CRT}(\mathcal{K})$, are defined in an analogous way using the operators of closed action calculi and the reflexive operator $\uparrow_p$ with the same arity rule as before. The equational theory CRAC is given by the set of equations upon closed reflexive terms generated by the action structure axioms A1–A6, the CAC axioms B1–B8 and the following additional axioms which account for the behaviour of $\uparrow_p$ in the closed setting:

$$R1: \quad \mathbf{id} = \uparrow_m \mathbf{i}_{m,m}$$

$$R2: \quad \uparrow_m t \otimes \mathbf{id} = \uparrow_m (t \otimes \mathbf{id})$$

$$R3: \quad \uparrow_m s \cdot t = \uparrow_m (s \cdot (\mathbf{id}_m \otimes t))$$

$$R4: \quad s \cdot \uparrow_m t = \uparrow_m ((\mathbf{id}_m \otimes s) \cdot t)$$

$$R5: \quad \uparrow_n \uparrow_m t = \uparrow_m \uparrow_n ((\mathbf{i}_{n,m} \otimes \mathbf{id}) \cdot t \cdot (\mathbf{i}_{m,n} \otimes \mathbf{id}))$$

$$R6: \quad \uparrow_{m \otimes n} t = \uparrow_n \uparrow_m t$$

We also call R1–R6 the reflexion axioms. It should be clear from the context whether the reflexion axioms refer to axioms in CRAC or RAC. Notice that we have chosen to define $\uparrow_m$ for arbitrary arities m, and include the axiom R6. An alternative approach would be to define reflexivity initially for the prime arities, remove axiom R6 and define the composite cases in terms of the prime cases and the other operators.

Given a reflexive action calculus $\mathrm{RAC}(\mathcal{K})$, the corresponding closed reflexive action calculus $\mathrm{CRAC}(\mathcal{K}_m)$ is defined in analogous way to definition 3.6, in the sense that $\mathcal{K}_m$ is the set defined in definition 3.6 and we require the additional control axioms on closed reflexive terms. We let $\mathrm{CRT}(\mathcal{K}_m)$ denote the set of closed reflexive terms generated by the control set $\mathcal{K}_m$.

The functions $[\![\_]\!]_{\vec{x}} : \mathrm{T}_{\{\vec{x}\}}(\mathcal{K}) \to \mathrm{CT}(\mathcal{K}_m)$ and $\langle \_ \rangle : \mathrm{CT}(\mathcal{K}_m) \to \mathrm{T}_{\emptyset}(\mathcal{K})$ are easily extended to account for this extra operator. We define the translations $[\![\_]\!]^r_{\vec{x}} : \mathrm{RT}_{\{\vec{x}\}}(\mathcal{K}) \to \mathrm{CRT}(\mathcal{K}_m)$ and $\langle \_ \rangle^r : \mathrm{CRT}(\mathcal{K}_m) \to \mathrm{RT}_{\emptyset}(\mathcal{K})$ by induction on the structure of terms as follows:

$$[\![\uparrow_p t]\!]^r_{\vec{x}} = \uparrow_p ((\mathbf{i}_{p,|\vec{x}|} \otimes \mathbf{id}) \cdot [\![t]\!]_{\vec{x}}),$$

$$\langle \uparrow_p t \rangle^r = \uparrow_p (\langle t \rangle),$$

and the other cases are the same as those given in definition 4.1 and 4.5 for $[\![\_]\!]_{\vec{x}}$ and $\langle \_ \rangle$ respectively. Analogous results to theorems 4.4, 4.6 and 4.7 hold.

Theorem 6.3

(i) $\{\vec{x}\} \vdash s = t \in \mathrm{RAC}_n$ implies $[\![s]\!]^r_{\vec{x}} = [\![t]\!]^r_{\vec{x}} \in \mathrm{CRAC}$.

(ii) $s = t \in \mathrm{CRAC}$ implies $\emptyset \vdash \langle s \rangle^r = \langle t \rangle^r \in \mathrm{RAC}_n$. □

Theorem 6.4 For $t \in \mathrm{RT}(\mathcal{K})$ and $s \in \mathrm{CRT}(\mathcal{K}_m)$, we have

(i) $\langle [\![t]\!]^r_{\vec{x}} \rangle^r = (\vec{x})t \in \mathrm{RAC}$, if $fn(t) \subseteq \{\vec{x}\}$.

(ii) $[\![\langle s \rangle^r]\!]^r_{\vec{x}} = \omega_{|\vec{x}|} \otimes s \in \mathrm{CRAC}$. □

Remark 6.5 In the appendix to [13], Jensen has argued for including the reflexion axiom

$$\rho 6: \quad \uparrow_m \mathbf{id}_m = \mathbf{id}_\varepsilon,$$

with the motivation that $\uparrow_m \mathbf{id}_m$ is inactive and inaccessible. If we incorporate the same axiom into CRAC, we obtain analogous results to theorem 6.3.

7 Concluding Remarks

We have introduced the notion of closed action calculi, and shown that there is a strong correspondence between the static parts of an arbitrary action calculus and its corresponding closed action calculus. This correspondence is given via a type theoretic presentation of action calculi, called the contextual action calculi, which give a local account of names using contexts. We have also shown that our ideas easily extend to Milner's reflexive action calculi. A general account connecting the dynamics of action calculi and closed action calculi is beyond the scope of this paper. We have shown the connection for the action calculi corresponding to the π-calculus and the λ-calculus.

Misfud, Milner and Power [6] have recently defined a category $\mathrm{CS}(\mathcal{K})$ of the so-called *control structures*, which provide models for the action calculus $\mathrm{AC}(\mathcal{K})$ such that $\mathrm{AC}(\mathcal{K})$ is initial in $\mathrm{CS}(\mathcal{K})$. Hermida and Power [5] and Power [15] have studied two name-free formulations of control structures, called the *fibrational* and *elementary* control structures respectively. One area for future research is to understand the link between their formulations, and the contextual action calculi and closed action calculi defined here.

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